

# F5C: a variant of Faugère's F5 algorithm with reduced Gröbner bases

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## Abstract

Faugère's F5 algorithm computes a Gröbner basis incrementally, by computing a sequence of (non-reduced) Gröbner bases. The authors describe a variant of F5, called F5C, that replaces each intermediate Gröbner basis with its reduced Gröbner basis. As a result, F5C considers fewer polynomials and performs substantially fewer polynomial reductions, so that it terminates more quickly. We also provide a generalization of Faugère's characterization theorem for Gröbner bases.

*Key words:* F5, Buchberger's Criteria, Reduced Gröbner Bases

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## 1. Introduction

Gröbner bases, first introduced in (Buchberger, 1965), are by now a fundamental tool of computational algebra, and Faugère's F5 algorithm is noted for its success at computing certain difficult Gröbner bases (Faugère, 2002; Bardet et al., 2003; Faugère, 2005). The algorithm's design is incremental: given a list of polynomials  $F = (f_1, \dots, f_m)$ , F5 computes for each  $i = 2, \dots, m$  a Gröbner basis  $G_i$  of the ideal  $\langle F_i \rangle = \langle f_1, \dots, f_i \rangle$  using a Gröbner basis  $G_{i-1}$  of the ideal  $\langle F_{i-1} \rangle$ . The algorithm assigns each polynomial  $p$  a "signature" determined by how it computed  $p$  from  $F$ ; using the signature, F5 detects a large number of zero reductions, and sometimes avoids these costly computations altogether.

This paper considers the challenge of modifying F5 so that it replaces  $G_{i-1}$  with its *reduced* Gröbner basis  $B_{i-1}$  before proceeding to  $\langle F_i \rangle$ . Working with the reduced Gröbner basis is desirable because each stage of the pseudocode of (Faugère, 2002) usually generates many polynomials that are not needed for the Gröbner basis property, and there is no interreduction between stages. In one example, we show that a straightforward implementation of the pseudocode of (Faugère, 2002) on Katsura-9 concludes with a Gröbner basis where nearly a third of the polynomials are unnecessary.

Stegers introduces a variant that uses  $B_{i-1}$  to *reduce* newly computed generators of  $\langle F_i \rangle$  (Stegers, 2006). We call this variant F5R, for "F5 Reducing by reduced Gröbner

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bases.” However, F5R still uses the unreduced basis  $G_{i-1}$  to compute critical pairs and new polynomials for  $G_i$ . As Stegers points out, discarding  $G_{i-1}$  in favor of  $B_{i-1}$  is not a casual task, since the signatures of  $G_{i-1}$  do not correspond to the polynomials of  $B_{i-1}$ .

The solution we propose is to generate new signatures that correspond to  $B_{i-1}$ , which generates the same ideal as  $F_{i-1}$ . With this change, we can discard  $G_{i-1}$  completely. The modified algorithm generates fewer polynomials and performs fewer reduction operations. Naturally, this means that the new variant consumes less CPU time, as documented in two different implementations. Although it is a non-trivial variant of F5, it respects its ancestor’s elegant structure and modifies only one subalgorithm. We call this variant F5C, for “F5 Computing by reduced Gröbner bases.”

After a review of preliminaries in Section 2, we describe F5C in Section 3, and provide some run-time data. A preliminary implementation in SINGULAR is complete (Greuel et al., 2005), and we present comparative timings for F5, F5R, and F5C. A proof of correctness appears in Section 4, and in Section 4.4 we show that one of Faugère’s criteria is a special case of a more general criterion.

The authors have made available a prototype implementation of F5, F5R, and F5C as a SINGULAR library (Greuel et al., 2005; Greuel and Pfister, 2008) at

[http://www.math.usm.edu/perry/Research/f5\\_library.lib](http://www.math.usm.edu/perry/Research/f5_library.lib).

A prototype implementation for the Sage computer algebra system (Stein, 2008) developed by Martin Albrecht, with some assistance from the authors, is available at

[http://bitbucket.org/malb/algebraic\\_attacks/src/tip/f5.py](http://bitbucket.org/malb/algebraic_attacks/src/tip/f5.py).

This latter implementation can use F4-style reduction.

## 2. Background Material

This section describes the fundamental notions and the conventions in this paper. Our conventions differ somewhat from Faugère’s, partly because the ones here make it relatively easy to describe and implement the variant F5C.

Let  $\mathbb{F}$  be a field and  $\mathcal{R} = \mathbb{F}[x_1, x_2, \dots, x_n]$ . Let  $<_T$  denote a fixed admissible ordering on the monomials  $\mathbb{M}$  of  $\mathcal{R}$ . For every polynomial  $p \in \mathcal{R}$  we denote the head monomial of  $p$  with respect to  $<_T$  by  $\text{HM}(p)$  and the head coefficient with respect to  $<_T$  by  $\text{HC}(p)$ . (For us, a monomial has no coefficient.) Let  $F = (f_1, f_2, \dots, f_m) \in \mathcal{R}^m$ . The goal of F5 is to compute a *Gröbner basis* of the ideal  $I = \langle F \rangle$  with respect to  $<_T$ .

### 2.1. Gröbner bases

A *Gröbner basis* of  $I$  with respect to  $<_T$  is a finite list  $G$  of polynomials in  $I$  that satisfies the properties  $\langle G \rangle = I$  and for every  $p \in I$  there exists  $g \in G$  satisfying  $\text{HM}(g) \mid \text{HM}(p)$ . Gröbner bases exist for any ideal of  $\mathcal{R}$ , and Buchberger first found an algorithm to compute such a basis (Buchberger, 1965). We can describe Buchberger’s algorithm in the following way: set  $G = F$ , then iterate the following three steps.

- Choose a *critical pair*  $p, q \in G$  that has not yet been considered, and construct its *S-polynomial*

$$S = \text{Spol}(p, q) = \text{HC}(q) \sigma_{p,q} \cdot p - \text{HC}(p) \sigma_{q,p} \cdot q$$

where

$$\sigma_{p,q} = \frac{\text{lcm}(\text{HM}(p), \text{HM}(q))}{\text{HM}(p)} \quad \text{and} \quad \sigma_{q,p} = \frac{\text{lcm}(\text{HM}(p), \text{HM}(q))}{\text{HM}(q)}.$$

We call  $p$  and  $q$  the *generators* of  $S$  and  $\sigma_{p,q} \cdot p$  and  $\sigma_{q,p} \cdot q$  the *components* of  $S$ .

- *Top-reduce*  $S$  with respect to  $G$ . That is, while  $t = \text{HM}(S)$  remains divisible by  $u = \text{HM}(g)$  for some  $g \in G$ , put  $S := S - \frac{\text{HC}(S)}{\text{HC}(g)} \frac{t}{u} \cdot g$ .
- Once no more top-reductions of  $S$  are possible, either  $S = 0$  or  $\text{HM}(S)$  is no longer divisible by  $\text{HM}(g)$  for any  $g \in G$ .
  - In the first case, we say that  $\text{Spol}(p, q)$  *reduces to zero with respect to*  $G$ .
  - In the second case, append  $S$  to  $G$ . The new entry in  $G$  means that  $\text{Spol}(p, q)$  now reduces to zero with respect to  $G$ .

The algorithm terminates once the  $S$ -polynomials of all pairs  $p, q \in G$  top-reduce to zero. That this occurs despite the introduction of new critical pairs when  $S$  does not reduce to zero is a well-known consequence of the Ascending Chain Condition (Becker et al., 1993; Cox et al., 1997).

In this paper we consider several kinds of *representations* of a polynomial. Let  $G$  and  $\mathbf{h}$  be lists of  $m$  elements of  $\mathcal{R}$ ,  $p \in \langle G \rangle$ , and  $t \in \mathbb{M}$ . We say that

- $\mathbf{h}$  is a *G-representation* of  $p$  if  $p = h_1 g_1 + \dots + h_m g_m$ ;
- $\mathbf{h}$  is a *t-representation of p with respect to G* if  $\mathbf{h}$  is a  $G$ -representation and for all  $k = 1, \dots, m$  we have  $h_k = 0$  or  $\text{HM}(h_k g_k) \leq_T t$ ; and
- $\mathbf{h}$  is an *S-representation of*  $S = \text{Spol}(g_i, g_j)$  *with respect to G* if  $\mathbf{h}$  is a  $t$ -representation of  $S$  with respect to  $G$  for some monomial  $t <_T \text{lcm}(\text{HM}(g_i), \text{HM}(g_j))$ .

We generally omit the phrase “with respect to  $G$ ” when it is clear from context.

If  $p$  top-reduces to zero with respect to  $G$ , then it is easy to derive an  $\text{HM}(p)$ -representation of  $p$ , although the converse is not always true. Correspondingly, if  $p$  is an  $S$ -polynomial and  $p$  top-reduces to zero with respect to  $G$ , then there exists an  $S$ -representation of  $p$ .

Theorem 1 summarizes three important characterizations of a Gröbner basis; (C) is from Buchberger (1965), while (D) is from Lazard (1983). The proof, and many more characterizations of a Gröbner basis, can be found in (Becker et al., 1993).

**Theorem 1.** *Let  $G$  be a finite list of polynomials in  $\mathcal{R}$ , and  $<_T$  an ordering on the monomials of  $\mathcal{R}$ . The following are equivalent:*

- (A)  $G$  is a Gröbner basis with respect to  $<_T$ .

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**Algorithm 1** BASIS

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```
1: globals  $r, Rule, <_T$ 
2: inputs
3:    $F = (f_1, f_2, \dots, f_m) \in \mathcal{R}^m$  (homogeneous)
4:    $<$ , an admissible ordering
5: outputs
6:   a Gröbner basis of  $F$  with respect to  $<$ 
7: do
8:    $<_T := <$ 
9:   Sort  $F$  by increasing total degree, breaking ties by increasing head monomial
   — Initialize the record keeping.
10:   $Rule := \text{List}(\text{List}())$ 
11:   $r := \text{List}()$ 
   — Compute the basis of  $\langle f_1 \rangle$ .
12:  Append  $(\mathbf{F}_1, f_1 \cdot \text{HC}(f_1)^{-1})$  to  $r$ 
13:   $G_{\text{prev}} = \{1\}$ 
14:   $B = \{f_1\}$ 
   — Compute the bases of  $\langle f_1, f_2 \rangle, \dots, \langle f_1, f_2, \dots, f_m \rangle$ .
15:   $i := 2$ 
16:  while  $i \leq m$ 
17:    Append  $(\mathbf{F}_i, f_i \cdot \text{HC}(f_i)^{-1})$  to  $r$ 
18:     $G_{\text{curr}} := \text{INCREMENTAL\_BASIS}(i, B, G_{\text{prev}})$ 
19:    if  $\exists \lambda \in G_{\text{curr}}$  such that  $\text{Poly}(\lambda) = 1$ 
20:      return  $\{1\}$ 
21:     $G_{\text{prev}} := G_{\text{curr}}$ 
22:     $B := \{\text{Poly}(\lambda) : \lambda \in G_{\text{prev}}\}$ 
23:     $i := i + 1$ 
24:  return  $B$ 
```

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(B) For all nonzero  $p \in \langle G \rangle$  there exists  $g \in G$  such that  $\text{HM}(g) \mid \text{HM}(p)$ .

(C) For all  $p, q \in G$   $\text{Spol}(p, q)$  top-reduces to zero with respect to  $G$ .

(D) For all  $p, q \in G$   $\text{Spol}(p, q)$  has an  $S$ -representation with respect to  $G$ .

## 2.2. The F5 Algorithm

In this section we give a brief overview of F5 (Algorithms 1–10). To make the presentation of F5R and F5C easier, we have made some minor modifications to the pseudocode of Faugère (2002); Stegers (2006), but they are essentially equivalent.

The F5 algorithm (Faugère, 2002) consists of several subalgorithms.

- The entry point is the BASIS. It expects as input a list of homogeneous polynomials of  $\mathcal{R}$ . BASIS invokes INCREMENTAL\_BASIS to construct Gröbner bases of the ideals  $\langle F_2 \rangle, \langle F_3 \rangle, \dots, \langle F_m \rangle$ , in succession. (Computing the Gröbner basis of  $\langle F_1 \rangle$  is trivial.) Polynomials are stored in a data structure  $r$ , whose details we consider in

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**Algorithm 2** INCREMENTAL\_BASIS

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```
1: globals  $r, <_T$ 
2: inputs
3:    $i \in \mathbb{N}$ 
4:    $B$ , a Gröbner basis of  $(f_1, f_2, \dots, f_{i-1})$  with respect to  $<_T$ 
5:    $G_{\text{prev}} \subset \mathbb{N}$ , indices in  $r$  of  $B$ 
6: outputs
7:    $G_{\text{curr}}$ , indices in  $r$  of a Gröbner basis of  $(f_1, f_2, \dots, f_i)$  with respect to  $<_T$ 
8: do
9:    $\text{curr\_idx} := \#r$ 
10:   $G_{\text{curr}} := G_{\text{prev}} \cup \{\text{curr\_idx}\}$ 
11:  Append List  $()$  to  $\text{Rule}$ 
12:   $P := \bigcup_{j \in G_{\text{prev}}} \text{CRITICAL\_PAIR}(\text{curr\_idx}, j, i, G_{\text{prev}})$ 
13:  while  $P \neq \emptyset$ 
14:     $d := \min \{\deg t : (t, k, u, \ell, v) \in P\}$  — See Algorithm 3 for structure of  $p \in P$ 
15:     $P_d := \{(t, k, u, \ell, v) \in P : d = \deg t\}$ 
16:     $P := P \setminus P_d$ 
17:     $S := \text{COMPUTE\_SPOLS}(P_d)$ 
18:     $R := \text{REDUCTION}(S, B, G_{\text{prev}}, G_{\text{curr}})$ 
19:    for  $k \in R$ 
20:       $P := P \cup \left( \bigcup_{j \in G_{\text{curr}}} \text{CRITICAL\_PAIR}(k, j, i, G_{\text{prev}}) \right)$ 
21:       $G_{\text{curr}} := G_{\text{curr}} \cup \{k\}$ 
22:  return  $G_{\text{curr}}$ 
```

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Section 2.3. The sets  $G_{\text{curr}}, G_{\text{prev}} \subset \mathbb{N}$  index elements of  $r$  that correspond to the generators of  $\langle F_i \rangle$  and a Gröbner basis of  $\langle F_{i-1} \rangle$ , respectively.

- The goal of INCREMENTAL\_BASIS is to compute a Gröbner basis of  $\langle F_i \rangle$  by computing  $d$ -Gröbner bases for  $d = 1, 2, \dots$  (A  $d$ -Gröbner basis is one for which all  $S$ -polynomials of homogeneous degree at most  $d$  reduce to zero; see (Becker et al., 1993).) INCREMENTAL\_BASIS iterates the following steps, which follow the general outline of Buchberger’s Algorithm:
  - Generate a list of critical pairs by iterating CRITICAL\_PAIR on all of the pairs of  $\{\text{curr\_idx}\} \times G_{\text{prev}}$ . (In our implementation,  $\text{curr\_idx}$  is the location in  $r$  where  $f_i$  is stored.)
  - Identify the critical pairs of smallest degree, and compute the necessary  $S$ -polynomials of smallest degree using COMPUTE\_SPOLS.
  - Top-reduce by passing the output  $S$  of COMPUTE\_SPOLS to REDUCTION.
  - The output  $R$  of REDUCTION indexes those polynomials that did not reduce to zero; new critical pairs are generated by iterating CRITICAL\_PAIR on all pairs  $(k, j) \in R \times G_{\text{curr}}$ , and  $R$  is appended to  $G_{\text{curr}}$ .

We highlight the major differences between these subalgorithms and their counterparts in Buchberger’s algorithm:

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**Algorithm 3** CRITICAL\_PAIR

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```
1: globals  $<_T$ 
2: inputs
3:    $k, \ell \in \mathbb{N}$  such that  $1 \leq k < \ell \leq \#r$ 
4:    $i \in \mathbb{N}$ 
5:    $G_{\text{prev}} \subset \mathbb{N}$ , indices in  $r$  of a Gröbner basis of  $(f_1, f_2, \dots, f_{i-1})$  w/respect to  $<_T$ 
6: outputs
7:    $\{(t, u, k, v, \ell)\}$ , corresponding to a critical pair  $\{k, \ell\}$  necessary for
8:     the computation of a Gröbner basis of  $(f_1, f_2, \dots, f_i)$ ;  $\emptyset$  otherwise
9: do
10:   $t_k := \text{HM}(\text{Poly}(k))$ 
11:   $t_\ell := \text{HM}(\text{Poly}(\ell))$ 
12:   $t := \text{lcm}(t_k, t_\ell)$ 
13:   $u_1 := t/t_k$ 
14:   $u_2 := t/t_\ell$ 
15:   $\tau_1 \mathbf{F}_{\nu_1} := \text{Sig}(k)$ 
16:   $\tau_2 \mathbf{F}_{\nu_2} := \text{Sig}(\ell)$ 
17:  if  $\nu_1 = i$  and  $u_1 \cdot \tau_1$  is top-reducible by  $G_{\text{prev}}$ 
18:    return  $\emptyset$ 
19:  if  $\nu_2 = i$  and  $u_2 \cdot \tau_2$  is top-reducible by  $G_{\text{prev}}$ 
20:    return  $\emptyset$ 
21:  if  $u_1 \cdot \text{Sig}(k) \prec u_2 \cdot \text{Sig}(\ell)$ 
22:    Swap  $u_1$  and  $u_2$ 
23:    Swap  $k$  and  $\ell$ 
24:  return  $\{(t, k, u_1, \ell, u_2)\}$ 
```

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**Algorithm 4** COMPUTE\_SPOLS

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```
1: globals  $r, <_T$ 
2: inputs
3:    $P$ , a set of critical pairs in the form  $(t, k, u, \ell, v)$ 
4: outputs
5:    $S$ , a list of indices in  $r$  of  $S$ -polynomials computed
6:   for a Gröbner basis of  $(f_1, f_2, \dots, f_i)$ 
7: do
8:    $S := ()$ 
9:   for  $(t, k, u, \ell, v) \in P$ , from smallest to largest lcm
10:    if not  $\text{Is\_REWRITABLE}(u, k)$  and not  $\text{Is\_REWRITABLE}(v, \ell)$ 
11:      Compute  $s$ , the  $S$ -polynomial of  $\text{Poly}(k)$  and  $\text{Poly}(\ell)$ 
12:      Append  $(u \cdot \text{Sig}(k), s)$  to  $r$ 
13:       $\text{ADD\_RULE}(u \cdot \text{Sig}(k), \#r)$ 
14:      if  $s \neq 0$ 
15:        Append  $\#r$  to  $S$ 
16:   Sort  $S$  by increasing signature
17:  return  $S$ 
```

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**Algorithm 5** REDUCTION

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```
1: globals  $r, <_T$ 
2: inputs
3:    $S$ , a list of indices of polynomials added to the generators  $G_i$ 
4:    $B$ , a Gröbner basis of  $(f_1, f_2, \dots, f_{i-1})$  with respect to  $<_T$ 
5:    $G_{\text{prev}} \subset \mathbb{N}$ , indices in  $r$  corresponding to  $B$ 
6:    $G_{\text{curr}} \subset \mathbb{N}$ , indices in  $r$  of a list of generators of the ideal of  $(f_1, f_2, \dots, f_i)$ 
7: outputs
8:    $\text{completed}$ , a subset of  $G$  corresponding to (mostly) top-reduced polynomials
9: do
10:   $to\_do := S$ 
11:   $\text{completed} := \emptyset$ 
12:  while  $to\_do \neq ()$ 
13:    Let  $k$  be the element of  $to\_do$  such that  $\text{Sig}(k)$  is minimal.
14:     $to\_do := to\_do \setminus \{k\}$ 
15:     $h := \text{Normal\_Form}(\text{Poly}(k), B, <_T)$ 
16:     $r_k := (\text{Sig}(k), h)$ 
17:     $\text{newly\_completed}, \text{redo} := \text{TOP\_REDUCTION}(k, G_{\text{prev}}, G_{\text{curr}} \cup \text{completed})$ 
18:     $\text{completed} := \text{completed} \cup \text{newly\_completed}$ 
19:    for  $j \in \text{redo}$ 
20:      Insert  $j$  in  $to\_do$ , sorting by increasing signature
21:  return  $\text{completed}$ 
```

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- CRITICAL\_PAIR discards any pair whose corresponding  $S$ -polynomial has a component that satisfies the “new criterion” of (Faugère, 2002), described in Section 4.4.
- COMPUTE\_SPOLS disregards any  $S$ -polynomial with a “rewritable” component, as described in Section 4.2.
- REDUCTION iterates over the most recently computed  $S$ -polynomials, from lowest signature to highest. For each  $k$  in its input, it:
  - Performs a complete (normal form) reduction of  $\text{Poly}(k)$  by the previous Gröbner basis.
  - Invokes TOP\_REDUCTION, which top-reduces  $\text{Poly}(k)$  by the current set of generators, subject to the following restrictions.
    - \* TOP\_REDUCTION invokes FIND\_REDUCATOR to find top-reductions. If it finds one, TOP\_REDUCTION may act in two different ways, depending on the signature of the top-reduction. If the signature is “safe”, which means “signature-preserving”, as discussed at the end of Section 4.1, then an ordinary top-reduction takes place. If the signature is “unsafe”, then TOP\_REDUCTION acts as if it is computing an  $S$ -polynomial, and thus generates a new polynomial with the new (higher) signature.
    - \* Some top-reductions by the current basis are forbidden by Line 16 of FIND\_REDUCATOR. The practical result is that some polynomials in the

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**Algorithm 6** TOP\_REDUCTION

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```
1: globals  $r, <_T$ 
2: inputs
3:    $k$ , the index of a labeled polynomial
4:    $G_{\text{prev}} \subset \mathbb{N}$ , indices in  $r$  of a Gröbner basis of  $(f_1, f_2, \dots, f_{i-1})$  w/respect to  $<_T$ 
5:    $G_{\text{curr}} \subset \mathbb{N}$ , indices in  $r$  of a list of generators of the ideal of  $(f_1, f_2, \dots, f_i)$ 
6: outputs
7:   completed, which has value  $\{k\}$  if  $r_k$  was not top-reduced and  $\emptyset$  otherwise
8:   to_do, which has value
9:      $\emptyset$  if  $r_k$  was not top-reduced,
10:     $\{k\}$  if  $r_k$  is replaced by its top-reduction, and
11:     $\{k, \#r\}$  if top-reduction of  $r_k$  generates a polynomial with a signature larger
    than  $\text{Sig}(k)$ .
12: do
13:   if  $\text{Poly}(k) = 0$ 
14:     warn "Reduction to zero!"
15:     return  $\emptyset, \emptyset$ 
16:    $p := \text{Poly}(k)$ 
17:    $J := \text{FIND\_REDUCTOR}(k, G_{\text{prev}}, G_{\text{curr}})$ 
18:   if  $J = \emptyset$ 
19:      $r_k := (\text{Sig}(k), p \cdot (\text{HC}(p))^{-1})$ 
20:     return  $\{k\}, \emptyset$ 
21:   Let  $j$  be the single element in  $J$ 
22:    $q := \text{Poly}(j)$ 
23:    $u := \frac{\text{HM}(p)}{\text{HM}(q)}$ 
24:    $c := \text{HC}(p) \cdot (\text{HC}(q))^{-1}$ 
25:    $p := p - c \cdot u \cdot q$ 
26:   if  $p \neq 0$ 
27:      $p := p \cdot (\text{HC}(p))^{-1}$ 
28:   if  $u \cdot \text{Sig}(j) \prec \text{Sig}(k)$ 
29:      $r_k := (\text{Sig}(k), p)$ 
30:     return  $\emptyset, \{k\}$ 
31:   else
32:     Append  $(u \cdot \text{Sig}(j), p)$  to  $r$ 
33:     ADD_RULE( $u \cdot \text{Sig}(j), \#r$ )
34:   return  $\emptyset, \{k, \#r\}$ 
```

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**Algorithm 7** FIND\_REDUCTOR

---

```
1: globals  $<_T$ 
2: inputs
3:    $k$ , the index of a labeled polynomial
4:    $G_{\text{prev}} \subset \mathbb{N}$ , indices in  $r$  of a Gröbner basis with respect to  $<_T$  of  $(f_1, f_2, \dots, f_{i-1})$ 
5:    $G_{\text{curr}} \subset \mathbb{N}$ , indices in  $r$  of a list of generators of the ideal of  $(f_1, f_2, \dots, f_i)$ 
6: outputs
7:    $J$ , where  $J = \{j\}$  if  $j \in G_{\text{curr}}$  and  $\text{Poly}(k)$  is safely top-reducible by  $\text{Poly}(j)$ ;
8:   otherwise  $J = \emptyset$ 
9: do
10:   $t := \text{HM}(\text{Poly}(k))$ 
11:  for  $j \in G_{\text{curr}}$ 
12:     $t' = \text{HM}(\text{Poly}(j))$ 
13:    if  $t' \mid t$ 
14:       $u := t/t'$ 
15:       $\tau_j \mathbf{F}_{\nu_j} := \text{Sig}(j)$ 
16:      if  $u \cdot \text{Sig}(j) \neq \text{Sig}(k)$  and not IS_REWRITABLE( $u, j$ ) and  $u \cdot \tau_j$  is not top-
        reducible by  $G_{\text{prev}}$ 
17:        return  $\{j\}$ 
18: return  $\emptyset$ 
```

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**Algorithm 8** ADD\_RULE

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```
1: globals  $r, \text{Rule}$ 
2: inputs
3:    $\tau \mathbf{F}_\nu$ , the signature of  $r_k$ 
4:    $k$ , the index of a labeled polynomial in  $r$  (or 0, for a phantom labeled polynomial)
5: do
6:   Append  $(\tau, k)$  to  $\text{Rule}_\nu$ 
7: return
```

---

basis may not be fully top-reduced. These correspond to forbidden  $S$ -polynomials; compare with lines 17 and 19 of CRITICAL\_PAIR and line 10 of COMPUTE\_SPOLS.

The remaining subalgorithms record and analyze information used by CRITICAL\_PAIR and COMPUTE\_SPOLS to discard useless pairs:

- ADD\_RULE is invoked whenever COMPUTE\_SPOLS or REDUCTION generates a new polynomial, and records information about that polynomial.
- IS\_REWRITABLE and FIND\_REWRITING determine when an  $S$ -polynomial is rewritable.

### 2.3. Signatures and Labeled Polynomials in F5

The first major difference between F5 and traditional algorithms to compute a Gröbner basis is the additional record keeping of “signatures”.

---

**Algorithm 9** Is\_REWRITABLE

---

```
1: inputs  
2:    $u$ , a power product  
3:    $k$ , the index of a labeled polynomial in  $r$   
4: outputs  
5:   true if  $u \cdot \text{Sig}(k)$  is rewritable (see FIND_REWRITING)  
6: do  
7:    $j := \text{FIND\_REWRITING}(u, k)$   
8:   return  $j \neq k$ 
```

---

---

**Algorithm 10** FIND\_REWRITING

---

```
1: globals  $Rule$   
2: inputs  
3:    $u$ , a power product  
4:    $k$ , the index of a labeled polynomial in  $r$   
5: outputs  
6:    $j$ , the index of a labeled polynomial in  $r$  such that if  $\tau_j \mathbf{F}_{\nu_j} = \text{Sig}(j)$   
       and  $\tau_j \mathbf{F}_{\nu_j} = \text{Sig}(k)$ , then  $\nu_j = \nu_k$  and  $\tau_j \mid u \cdot \tau_k$   
       and  $r_j$  was added to  $Rule_{\nu_k}$  most recently.  
7: do  
8:    $\tau_k \mathbf{F}_{\nu} := \text{Sig}(k)$   
9:    $ctr := \#Rule_{\nu}$   
10:  while  $ctr > 0$   
11:     $(\tau_j, j) := Rule_{\nu, ctr}$   
12:    if  $\tau_j \mid u \cdot \tau_k$   
13:      return  $j$   
14:     $ctr := ctr - 1$   
15:  return  $k$ 
```

---

**Definition 2.** Let  $M \in \mathbb{N}$ ,  $G = (g_1, \dots, g_M) \in \mathcal{R}^M$ , and  $p \in \mathcal{R}$ . We say that  $(\tau, \nu) \in \mathbb{M} \times \mathbb{N}$  is a *signature of  $p$  with respect to  $G$*  if  $p$  has an  $G$ -representation  $\mathbf{h}$  such that

- $h_{\nu+1} = h_{\nu+2} = \dots = h_M = 0$ ; and
- $\tau = \text{HM}(h_\nu)$ .

We omit the phrase “with respect to  $G$ ” when it is clear from context, and let  $\tau\mathbf{F}_\nu$  be a shorthand for  $(\tau, \nu)$ . We also say that  $\mathbf{h}$  is a  *$G$ -representation of  $p$  corresponding to  $\tau\mathbf{F}_\nu$* . We call  $\nu$  the *index*.

We also define the *zero signature*  $\mathbf{0}$  of the zero polynomial  $0g_1 + 0g_2 + \dots + 0g_M$ .

The *labeled polynomial*  $r_k = (\text{Sig}(k), \text{Poly}(k))$  is *admissible with respect to  $G$*  if  $\text{Sig}(k)$  is a signature of  $\text{Poly}(k)$  with respect to  $G$ . Again, we omit the phrase “with respect to  $G$ ” when it is clear from context.

*Remark.* Our definitions of a signature differ from Faugère’s in several respects:

- The first is minor: we use  $f_{\nu+1} = \dots = f_m = 0$  whereas Faugère uses  $f_1 = \dots = f_{\nu-1} = 0$ . The present version simplifies considerably the description and implementation of F5C.
- Faugère uses  $(\mathbf{F}_1, \dots, \mathbf{F}_m)$  as the basis for the  $\mathcal{R}$ -module  $\mathcal{R}^m$  where  $m$  is fixed; in F5C  $m$  usually increases.
- Faugère’s definition admits only one unique signature per polynomial, determined by a minimality criterion. Our version allows a polynomial to have many signatures; we refer to Faugère’s signature as the *minimal* signature of a polynomial. The change is motivated by a desire to reflect the algorithm’s behavior; for many inputs, F5 does not always assign the minimal signature to a polynomial.
- We introduce a zero signature.

The algorithm’s behavior depends crucially on the assumption that all the elements of  $r$  are admissible. We show that the algorithm satisfies this property in Proposition 7.

**Example 3.** Suppose that  $F = (xy + x, y^2 - 1)$ . Then  $(\mathbf{F}_1, f_1)$  is admissible with respect to  $F$ . So is  $(x\mathbf{F}_2, f_1)$ , since  $f_1 = yf_1 - xf_2$ .  $\diamond$

It will be convenient at times to multiply monomials to signatures; thus for any monomial  $u$  and any  $k \in \{1, \dots, \#r\}$  we write the *natural signature of the product of  $u$  and  $\text{Sig}(k)$*  as

$$u\text{Sig}(k) = u \cdot \tau\mathbf{F}_\nu = (u\tau)\mathbf{F}_\nu.$$

If  $\tau\mathbf{F}_\nu$  is a signature of a polynomial  $p$ , then the natural signature of the product of  $u$  and  $\tau\mathbf{F}_\nu$  is a signature of  $up$ . For more properties of signatures, see Proposition 7 in Section 4.1.

We now generalize the ordering  $<_\tau$  to an ordering on signatures.

**Definition 4.** Let  $\mathcal{S}$  be the set of all possible signatures with respect to  $F$ . Define a relation  $\prec$  on  $\mathcal{S}$  in the following way: for all monomials  $\tau, \tau' \in \mathbb{M}$

- $\mathbf{0}$  is smaller than any other signature, and

- for all  $i, j \in \mathbb{N}$   $\tau' \mathbf{F}_i \prec \tau \mathbf{F}_j$  iff
  - $i < j$ , or
  - $i = j$  and  $\tau' <_T \tau$ .

It is clear that  $\prec$  is a well-ordering on  $\mathcal{S}$ , which implies that every polynomial has a minimal signature.

**Example 5.** In Example 3,  $\mathbf{F}_1$  is the minimal signature of  $f_1$  with respect to  $F$ .  $\diamond$

### 3. F5C: F5 Computing with reduced Gröbner bases

It turns out that F5 often generates many “redundant” polynomials. For the purposes of this discussion, a *redundant polynomial in a Gröbner basis*  $B$  is a polynomial  $p \in B$  whose head monomial is divisible by the head monomial of some  $q \in B \setminus \{p\}$ . It is obvious from (B) of Theorem 1 that  $p$  is unnecessary for the Gröbner basis property, and can be discarded. In the Example given in (Faugère, 2002)  $r_{10}$ , which has head monomial  $y^6 t^2$ , is a redundant polynomial because of  $r_8$ , which has head monomial  $y^5 t^2$ .

Why does this happen? A glance at line 16 of `FIND_REDUCATOR` reveals that some top-reductions are forbidden! Thus, despite the fact that it is often much, much faster than other algorithms, F5 still generates many redundant polynomials. Paradoxically, we cannot discard such polynomials safely before the algorithm has computed a Gröbner basis, because the unnecessary polynomials are marked with signatures that are necessary for the algorithm’s stability and correctness.

#### 3.1. Introducing F5C

Stegers introduces a limited use of reduced Gröbner bases to F5: variant F5R top-reduces by the polynomials of a reduced basis, but continues to compute critical pairs and  $S$ -polynomials with the polynomials of the unreduced basis. One can implement this relatively easily by changing line 22 of `BASIS` to

22     Let  $B$  be the interreduction of  $\{\text{Poly}(\lambda) : \lambda \in G_{\text{prev}}\}$

When we say “interreduction”, we also mean to multiply so that the head coefficient is unity; thus  $B$  is the unique reduced Gröbner basis of  $\langle F_i \rangle$ . Subsequently, `REDUCTION` will reduce  $\text{Poly}(k)$  completely by the interreduced  $B$ ; this does not affect the algorithm’s correctness because the signature of every polynomial in  $\langle B \rangle$  is smaller than the signature of any polynomial generated with  $f_i$ .

Why does F5R only top-reduce by the reduced basis, but not compute critical pairs and  $S$ -polynomials using the reduced basis? The algorithm needs signatures and polynomials to correspond, but the signatures of the polynomials of  $B$  are *unknown*. Merely replacing the polynomials indexed by  $G_{\text{prev}}$  to those of  $B$  would render most polynomials inadmissible. The rewritings stored in *Rule* would no longer correspond to the signatures of  $S$ -polynomials, so `IS_REWRITABLE` would reject some  $S$ -polynomials wrongly, and would fail to reject some  $S$ -polynomials when it should.

Can we get around this? In fact, we can: modify the lists  $r$  and *Rule* so that the polynomials of  $B$  are admissible, and the rewrite rules valid, with respect to  $\langle B \rangle =$

$\langle F_i \rangle$ . Suppose that `INCREMENTAL_BASIS` has terminated with value  $G_{\text{prev}}$  in `BASIS`. As in `F5R`, modify Line 22 of `BASIS` to interreduce  $\{\text{Poly}(\lambda) : \lambda \in G_{\text{prev}}\}$  and obtain the reduced Gröbner basis  $B$ . The next stage of the algorithm requires the computation of a Gröbner basis of  $\langle F_{i+1} \rangle$ . Certainly  $\langle F_{i+1} \rangle = \langle B \cup \{f_{i+1}\} \rangle$ . Reset  $r$  and *Rule*, then create new lists to reflect the signatures and rewritings for the corresponding  $B$ -representation:

- $r := ((\mathbf{F}_j, B_j))_{j=1}^{\#B}$ ; and
- for each  $j = 2, \dots, \#B$  and for each  $k = 1, \dots, j-1$  set  $\text{Rule}_j := (\sigma_{p,q}, 0)_{k=1}^{j-1}$  where  $p = B_j$  and  $q = B_k$ .

The first statement assigns signatures appropriate for the module  $\mathcal{R}^{\#F'}$ ; the second recreates the list of rewritings to reflect that the  $S$ -polynomials of  $B$  all reduce to zero. The redirection is to a non-existent polynomial  $r_0$ , which serves as a convenient, fictional *phantom polynomial*; one might say  $\text{Poly}(0) = 0$ . This reconstruction of  $r$  and *Rule* allows the algorithm to avoid needless reductions. (It turns out that the reconstruction of *Rule* is unnecessary. However, this is not obvious, so we leave the step in for the time being, and discuss this in Section 4.5.) We have now rewritten the original problem in an equivalent form, based on new information.

Although we address correctness in Section 4.5, let us consider for a moment the intuitive reason that this phantom polynomial  $r_0$  poses no difficulty to correctness. In the original `F5` algorithm, every  $S$ -polynomial generates a new polynomial in  $r$  and a corresponding rule in *Rule*. (See lines 14 and 15 of `COMPUTE_SPOLs`, lines 13–15 of `TOP_REDUCTION`, and lines 19 and 20 of `REDUCTION`.) If  $r_k$  reduces to zero for some  $k$ , then  $k$  is not added to  $G_{\text{curr}}$ , but the rewrite rule  $(\text{Sig}(k), k)$  remains in *Rule*. Thus the algorithm never uses  $\text{Poly}(k)$  again; however, it uses  $\text{Sig}(k)$  to avoid computing other polynomials with the same signature. The change we propose has the same effect on  $S$ -polynomials of  $B$ : we know *a priori* that they reduce to zero. We could add a large number of entries  $(\text{Sig}(k), 0)$  to  $r$ , but since the algorithm never uses them we would merely waste space. Instead, we redirect the signature  $\text{Sig}(k)$  to a phantom polynomial  $r_0$ , which like  $r_k$  is never in fact used.

We call the resulting algorithm `F5C`, and summarize the modifications in the pseudocode of Algorithms 11 and 12; the first replaces Algorithm 1 entirely. We have separated most of the modification of `BASIS` into `SETUP_REDUCED_BASIS`, a separate subalgorithm invoked by `BASIS/C`, the replacement for `BASIS`.

### 3.2. Experimental results

One way to compare the three variants would be to measure the absolute timings when computing various benchmark systems. By this metric, `F5R` generally outperforms `F5`, and `F5C` generally outperforms `F5R`: the exceptions are all toy systems, where the overhead of repeated interreduction and `SETUP_REDUCED_BASIS` outweigh the benefit of using a reduced Gröbner basis. Tables 1 and 2 give timings and ratios for the variants in two different implementations.

- Table 1 gives the results from an implementation written in Python for the Sage computer algebra system, version 3.4. Sage is built on several other systems, one of which is `SINGULAR 3-0-4`. Sage calls `SINGULAR` to perform certain operations, so some parts of the implementation run in compiled code, but most of the algorithm is

---

**Algorithm 11** BASIS/C

---

```
1: globals  $r, Rule, <_T$ 
2: inputs
3:    $F = (f_1, f_2, \dots, f_m) \in \mathcal{R}^m$  (homogeneous)
4:    $<$ , an admissible ordering
5: outputs
6:   a Gröbner basis of  $F$  with respect to  $<$ 
7: do
8:    $<_T := <$ 
9:   Sort  $F$  by increasing total degree, breaking ties by increasing leading monomial
10:   $Rule := \text{List}(\text{List}())$ 
11:   $r := \text{List}()$ 
12:  Append  $(\mathbf{F}_1, f_1 \cdot \text{HC}(f_1)^{-1})$  to  $r$ 
13:   $G_{\text{prev}} = \{1\}$ 
14:   $B = \{f_1\}$ 
15:   $i := 2$ 
16:  while  $i \leq m$ 
17:    Append  $(\mathbf{F}_{\#r+1}, f_i \cdot \text{HC}(f_i)^{-1})$  to  $r$ 
18:     $G_{\text{curr}} := \text{INCREMENTAL\_BASIS}(\#r, B, G_{\text{prev}})$ 
19:    if  $\exists \lambda \in G_{\text{curr}}$  such that  $\text{Poly}(\lambda) = 1$ 
20:      return  $\{1\}$ 
    — The only change to BASIS is the addition of this line
21:     $G_{\text{prev}} := \text{SETUP\_REDUCED\_BASIS}(G_{\text{curr}})$ 
22:     $B := \{\text{Poly}(\lambda) : \lambda \in G_{\text{prev}}\}$ 
23:     $i := i + 1$ 
24: return  $B$ 
```

---

system	F5 (sec)	F5R (sec)	F5C (sec)	F5R/F5	F5C/F5
Katsura-7	6.60	5.09	4.23	0.77	0.64
Katsura-8	111.05	52.22	43.88	0.47	0.40
Katsura-9	5577	1421	1228	0.25	0.22
Cyclic-6	3.91	3.88	3.41	0.99	0.87
Cyclic-7	1182	505	381	0.43	0.32
Cyclic-8	>4 days	231455	188497	N/A	N/A

Table 1: Ratios of timings in the Sage (Python) implementation

All timings obtained using the `cputime()` function in a Python implementation in Sage 3.2.1, on a computer with a 2.66GHz Intel Core 2 Quad (Q9450) running Ubuntu Linux with 3GB RAM. The ground field has characteristic 32003. \*Computation of Cyclic-8 in F5 has not terminated on the sixth day of computation, when this draft was committed. On other computers, the timing was comparable.

---

**Algorithm 12** SETUP\_REduced\_BASIS

---

```
1: globals  $r, Rule, <_T$ 
   (modifies  $r$  and  $Rule$ )
2: inputs
3:    $G_{\text{prev}}$ , a list of indices of polynomials in  $r$  that correspond to a Gröbner basis of
    $(f_1, \dots, f_i)$ 
4: outputs
5:    $G_{\text{curr}} \subset \mathbb{N}$ , indices of polynomials in  $r$  that correspond to a reduced Gröbner basis
   of  $(f_1, \dots, f_i)$ 
6: do
7:   Let  $B$  be the interreduction of  $\{\text{Poly}(k) : k \in G_{\text{prev}}\}$ 
8:    $G_{\text{curr}} := \{j\}_{j=1}^{\#B}$ 
9:    $r := \text{List}(\{(\mathbf{F}_j, B_j)\}_{j=1}^{\#B})$ 
   — Lemma 32 implies that lines 10–15 are unnecessary
   — All the  $S$ -polynomials of  $B$  reduce to zero; document this
10:   $Rule = \text{List}(\{\text{List}()\}_{j=1}^{\#B})$ 
11:  for  $j := 1$  to  $\#B - 1$ 
12:     $t := \text{HM}(B_j)$ 
13:    for  $k := j + 1$  to  $\#B$ 
14:       $u := \text{lcm}(t, \text{HM}(B_k)) / \text{HM}(B_k)$ 
15:      ADD_RULE( $u\mathbf{F}_k, 0$ )
16: return  $G_{\text{curr}}$ 
```

---

system	F5 (sec)	F5R (sec)	F5C (sec)	F5R/F5	F5C/F5
Katsura-7	0.30	0.34	0.31	1.13	1.03
Katsura-8	4.05	4.41	3.33	1.09	0.82
Katsura-9	127.14	142.81	82.48	1.12	0.65
Schrans-Troost	25.43	21.74	21.43	0.85	0.84
F633	0.34	0.40	0.30	1.18	0.88
F744	1252	1132	1075	0.90	0.86
Cyclic-6	.04	.03	.03	0.75	0.75
Cyclic-7	6.5	5.39	4.35	0.83	0.67
Cyclic-8	3233	3101	2154	0.96	0.67

Table 2: Timings for the (compiled) SINGULAR implementations

Average of four timings obtained from the `getTimer()` function in a modified SINGULAR 3-1-0 kernel, on a computer with a 3.16GHz Intel Xeon (X5460) running Gentoo Linux with 64GB RAM. The ground field has characteristic 32003.

system	reductions in F5	reductions in F5R	reductions in F5C
Katsura-4	774	289	222
Katsura-5	14597	5355	3985
Katsura-6	1029614	77756	58082
Cyclic-5	510	506	446
Cyclic-6	41333	23780	14167

Table 3: Reductions performed by the three F5 variants over a field of characteristic 32003.

otherwise implemented in Python. For example, Line 15 of REDUCTION (reduction by the previous basis) is handed off to SINGULAR, while the implementation of TOP\_REDUCTION is nearly entirely Python.

- Table 2 gives the results from a compiled SINGULAR implementation built on the SINGULAR 3-1 kernel. This implementation is unsurprisingly much, much faster than the Sage implementation. Nevertheless, the implementation is still a work in progress, lacking a large number of optimizations. For example, so far polynomials are represented by geobuckets (Yap, 2000); the eventual goal is to implement the F4-style reduction that Faugère advises for efficiency (Faugère, 1999, 2002).

*Remark.* This SINGULAR implementation has one major difference from the pseudocode given: its implementation of TOP\_REDUCTION performs safe reductions of non-leading monomials as well as of the leading monomials. This helps explain why there seems to be no benefit to F5R, unlike the Sage implementation. Another factor is that top-reduction in Sage is performed by interpreted Python code, whereas tail reductions are performed by the compiled SINGULAR library to which Sage links. Thus, the penalty for interreduction, relative to top-reduction, is much lower in Sage, to the benefit of F5R.

Timings alone are an unsatisfactory metric for this comparison. They depend heavily on the efficiency of hidden algorithms, such as the choice of polynomial representation (lists, buckets, sparse matrices). It is well-known that the most time-consuming part by far of any non-trivial Gröbner basis computation consists in the reduction operations: top-reduction, inter-reduction, and computing normal forms. This remains true for F5, with the additional wrinkle that, as mentioned before, F5 generally computes many more polynomials than are necessary for the Gröbner basis. Thus a more accurate comparison between the three variants would consider

- the number of critical pairs considered,
- the number of polynomials generated, and
- the number of reduction operations performed.

We present a few examples with benchmark systems in Tables 3–5, generated from the prototype implementation in Sage. In each case, the number of reductions performed by F5C remains substantially lower than the number performed by F5R, which is itself drastically lower than the number performed by F5. As a reference for comparison, we modified the toy implementation of the Gebauer-Möller algorithm that is included with the Sage computer algebra system to count all the reduction operations (Gebauer and



$i$	$\#G_{\text{curr}}$	$\max\{d\}$	$\max\{\#P_d\}$
2	2	N/A	N/A
3	4	3	$\#P_3 = 1$
4	8	4	$\#P_3 = 2$
5	16	6	$\#P_4 = \#P_5 = 4$
6	32	6	$\#P_4 = 8$
7	60	10	$\#P_5 = 17$
8	132	11	$\#P_6 = 29$
9	524	16	$\#P_8 = 89$
10	1165	13	$\#P_8 = 276$

Table 4: Internal data of INCREMENTAL\_BASIS in both F5 and F5R while computing Katsura-9.

$i$	$\#G_{\text{curr}}$	$\max\{d\}$	$\max\{\#P_d\}$
2	2	N/A	N/A
3	4	3	$\#P_3 = 1$
4	8	4	$\#P_3 = 2$
5	15	6	$\#P_3 = \#P_4 = 4$
6	29	6	$\#P_4 = \#P_6 = 6$
7	51	10	$\#P_5 = 12$
8	109	11	$\#P_6 = 29$
9	472	16	$\#P_8 = 71$
10	778	13	$\#P_8 = 89$

Table 5: Internal data of INCREMENTAL\_BASIS/C in F5C while computing Katsura-9.

Möller, 1988); it performed more than 1,500,000 reductions to compute Cyclic-6. The table shows that F5 performed approximately 2.4% of that number, while F5C performed approximately 0.7% of that number.

In general, F5 and F5R will compute the same number of critical pairs and polynomials, because they are using the same values of  $G_{\text{prev}}$ . Top-reducing by a reduced Gröbner basis eliminates the vast majority of reductions, but in F5R  $G_{\text{prev}}$  still indexes polynomials whose monomials are reducible by other polynomials, *including head monomials!* As a consequence, F5R cannot consider fewer critical pairs or generate fewer polynomials than F5. By contrast, F5C has discarded from  $G_{\text{prev}}$  polynomials with redundant head monomials, and has eliminated reducible lower order monomials. Correspondingly, there is less work to do.

**Example 6.** In the Katsura-9 system for F5 and F5R, each pass through the **while** loop of INCREMENTAL\_BASIS generates the internal data shown in Table 4. For F5C, each pass through the **while** loop of INCREMENTAL\_BASIS/C generates the internal data shown in Table 5. For each  $i$ , F5R and F5C both compute  $B$ , the unique reduced Gröbner basis of  $F_i$ . This significantly speeds up top-reduction, but F5C replaces  $r$  with labeled polynomials for  $B$ . The consequence is that  $G_{\text{prev}}$  contains fewer elements, leading INCREMENTAL\_BASIS/C to generate fewer critical pairs, and hence fewer polynomials for  $G_{\text{curr}}$ . Similar behavior occurs in other large systems.  $\diamond$

#### 4. Correctness of the output of F5 and F5C

In this section we prove that if F5 and F5C terminate, then their output is correct. Seeing that Faugère has already proved the correctness of F5, why do we include a new proof? First, we rely on certain aspects of the proof to explain the modifications that led to F5C, so it is convenient to re-present a proof here. Another reason is to present a new generalization of Faugère's characterization of a Gröbner basis; although it is not necessary for F5C, the new characterization is interesting enough to describe here.

*Remark.* We do not address the details of termination, nor will we even assert that the algorithms *do* terminate, but in practice we have not encountered any systems that do not terminate in F5.

Having said that, we would like to address an issue with which some readers may be familiar. The Magma source code of (Stegers, 2006) implements F5R. This code is publicly available, and contains an example system in the file `nonTerminatingExample.mag`. As the reader might expect from the name, this system causes an infinite loop when given as input to the source code. Roger Dellaca, Justin Gash, and John Perry traced this loop to an error in `TOP_REDUCTION`. (Lines 32 and 33 were not implemented, which sabotages the record-keeping of *Rule*.) The corrected Magma code terminates with the Gröbner basis of that system.

##### 4.1. Properties of signatures

The primary tool in F5 is the signature of a polynomial (Definition 2). The following properties of signatures explain certain choices made by the algorithm.

**Proposition 7.** *Let  $p, q \in \mathcal{R}$ ,  $\tau, \tau', u, v \in \mathbb{M}$ , and  $\nu, \nu' \in \{1, 2, \dots, M\}$ . Suppose that  $\tau \mathbf{F}_\nu$  and  $\tau' \mathbf{F}_{\nu'}$  are signatures of  $p$  and  $q$ , respectively. Each of the following holds:*

- (A)  $(u\tau) \mathbf{F}_\nu$  is a signature of  $up$ .
- (B) If  $u\tau \mathbf{F}_\nu \succ \tau' \mathbf{F}_{\nu'}$ , then  $(u\tau) \mathbf{F}_\nu$  is a signature of  $up \pm vq$ .
- (C) If  $(\sigma_{p,q}\tau) \mathbf{F}_\nu \succ (\sigma_{q,p}\tau') \mathbf{F}_{\nu'}$ , then  $(\sigma_{p,q}\tau) \mathbf{F}_\nu$  is a signature of  $\text{Spol}(p, q)$ .

The proof is straightforward, so we omit it.

The following proposition implies that the labeled polynomials of  $r$  are admissible with respect to the input at every moment during the algorithm's execution.

**Proposition 8.** *Each of the following holds.*

- (A) For every  $k \in \{1, 2, \dots, \#r\}$ ,  $\text{Sig}(k)$  is a signature of  $\text{Poly}(k)$  with respect to  $F$  when  $r_k$  is defined in Line 12 of `COMPUTE_SPOLS` and Line 32 of `TOP_REDUCTION`.
- (B) After the call

$$h := \text{Normal\_Form}(\text{Poly}(k), \text{safe}, <_T)$$

in Line 15 of `REDUCTION`,  $\text{Sig}(k)$  is a signature of  $h$  with respect to  $F$ .

- (C) For all  $k \in \{1, 2, \dots, \#r\}$ ,  $\text{Sig}(k)$  remains invariant, and is a signature of  $\text{Poly}(k)$  with respect to  $F$ .

The proof follows without difficulty from Proposition 7 and inspection of the algorithms that create or modify labeled polynomials: INCREMENTAL\_BASIS, COMPUTE\_SPOLS, REDUCTION, and TOP\_REDUCTION.

*Remark.* Although  $\text{Sig}(k)$  is a signature of  $\text{Poly}(k)$ , it need not be the *minimal* signature of  $\text{Poly}(k)$ . For example, if F5 is given the input  $F = (xh + h^2, yh + h^2)$  then COMPUTE\_SPOLS computes an  $S$ -polynomial and creates the labeled polynomial

$$r_3 = (x\mathbf{F}_2, yh^2 - xh^2).$$

Hence  $\text{Sig}(3) = x\mathbf{F}_2$ , but it is also true that

$$\text{Spol}(f_1, f_2) = -hf_1 + hf_2.$$

Thus  $h\mathbf{F}_2$  is also a signature of  $\text{Poly}(3)$ . Since  $h\mathbf{F}_2 \prec x\mathbf{F}_2$ ,  $x\mathbf{F}_2$  is not the minimal signature of  $f_2$ .

**Definition 9.** Let  $F \in \mathcal{R}^m$ ; all signatures are with respect to  $F$ . Suppose that  $\tau\mathbf{F}_\nu$  is a signature of an  $S$ -polynomial  $S$  generated by  $\text{Poly}(a)$  and  $\text{Poly}(b)$ , and  $\mathbf{h}$  is an  $S$ -representation of  $S$  such that the natural signatures of the products satisfy

$$\text{HM}(h_\lambda) \text{Sig}(\lambda) \prec \tau\mathbf{F}_\nu$$

for all  $\forall \lambda = 1, \dots, \#\mathbf{h}$  except one, say  $\lambda'$ , in which case  $\text{HM}(h_{\lambda'}) \text{Sig}(\lambda') = \tau\mathbf{F}_\nu$  and  $\lambda' > a, b$ . We call  $\mathbf{h}$  a *signature-preserving*  $S$ -representation.

Proposition 8 implies that top-reductions that do not generate new polynomials create signature-preserving  $S$ -representations of  $S$ -polynomials. Top-reductions that do generate new polynomials correspond to new  $S$ -polynomials, and the reductions of the new polynomials likewise correspond to signature-preserving  $S$ -representations. That motivates the following definition.

**Definition 10.** If we are at a stage of the algorithm where COMPUTE\_SPOLS generated  $r_k$ , but REDUCTION has not yet reduced it, we say that REDUCTION *is scheduled to compute a signature-preserving  $S$ -representation*. Once it computes the representation, we say that the algorithm *has computed a signature-preserving reduction to zero*.

#### 4.2. Rewritable Polynomials

As Faugère illustrates in Section 2 of (Faugère, 2002), linear algebra suggests that two rows of the Sylvester matrix of  $F$  need not be triangularized if one row has already been used in the triangularization of another row. This carries over into the  $F$ -representations of  $S$ -polynomial components, so F5 uses signatures to hunt for such redundant components. The structure *Rule* tracks which signatures have already been computed.

**Definition 11.** Let *Rule* be a list of  $m$  lists of tuples of the form  $\rho = (\tau, j)$ . We write  $\text{Rule}_i$  for the  $i$ th list in *Rule*. We say that *Rule* is a *list of rewritings for  $\mathbf{r}$*  if for every  $i = 1, \dots, m$  and for every  $\rho_\ell = (\tau, j) \in \text{Rule}_i$  there exist  $p, q \in \mathcal{R}$  such that

1.  $p = \text{Poly}(a)$ ,  $q = \text{Poly}(b)$  for some  $a, b \in G_{\text{curr}}$ ;
2.  $\max_{\prec} \{\sigma_{p,q} \cdot \text{Sig}(a), \sigma_{q,p} \cdot \text{Sig}(b)\} = \tau\mathbf{F}_i$ ;

3.  $j > a, b$  and the first defined value of  $\text{Poly}(j)$  is  $\text{Spol}(p, q)$ ;
4. there exists (or REDUCTION is scheduled to compute) a signature-preserving  $S$ -representation  $\mathbf{h}$  of  $\text{Spol}(p, q)$  such that  $h_j = 1$ ; and
5. if  $\rho_{\ell'} = (\tau', j') \in \text{Rule}_i$  and  $\ell' > \ell$ , then  $j' > j$ .

We call  $\text{Poly}(j)$  the *rewriting* of  $\text{Spol}(p, q)$ .

*Remark.* When we speak of  $\text{Spol}(p, q)$ , we include any unsafe top-reduction that is computed in TOP\_REDUCTION.

**Proposition 12.** *Every signature-preserving reduction by F5 of an  $S$ -polynomial  $S$  to the polynomial  $p$  (where possibly  $p = 0$ ) is recorded in some  $\text{Rule}_i$  by the entry  $(u \cdot \text{Sig}(k), j)$  where:*

- $S = u \cdot \text{Poly}(k) - v \cdot \text{Poly}(\ell)$  for some  $\ell \in G_{\text{curr}}$  and appropriate  $u, v \in \mathbb{M}$ ;
- $u \cdot \text{Sig}(k) \succ v \cdot \text{Sig}(\ell)$ ;
- the first defined value of  $\text{Poly}(j)$  is  $S$ , and the final value of  $\text{Poly}(j)$  is  $p$ ; and
- $j > k, \ell$ .

The proof follows from inspection of the algorithms that create and top-reduce polynomials.

**Proposition 13.** *At every point during the execution of F5, the global variable  $\text{Rule}$  satisfies Definition 11.*

The proof follows from Proposition 12 and inspection of the algorithms that create and modify  $\text{Rule}$ .

**Definition 14.** Let  $j, k \in G_{\text{curr}}$ ,  $u \in \mathbb{M}$ , and  $\text{Sig}(k) = \tau \mathbf{F}_\nu$ . At any given point during the execution of the algorithm we say that the polynomial multiple  $u \text{Poly}(k)$  is *rewritable by  $\text{Poly}(j)$  in  $W = \text{Rule}_\nu$*  if

- $k \neq j$ ;
- $\text{Poly}(j)$  is the rewriting of an  $S$ -polynomial;
- $\text{Sig}(j) = \tau' \mathbf{F}_\nu$  and  $\tau' \mid u\tau$  (note the same index  $\nu$  as  $\text{Sig}(k)$ );
- $(\tau', j) = W_a$  for some  $a \in \mathbb{N}$ ; and
- for any  $W_b = (\tau'', c)$  such that  $\tau'' \mid u\tau$ , either  $W_a = W_b$  or  $b < a$ .

We usually omit some or all of the phrase “by  $\text{Poly}(j)$  in  $\text{Rule}_\nu$ .” We call  $\text{Poly}(j)$  the *rewriter* of  $u \text{Poly}(k)$ .

**Proposition 15.** *Let  $u \in \mathbb{M}$  and  $k \in G_{\text{curr}}$ . The following are equivalent.*

- (A)  $u \text{Poly}(k)$  is rewritable in  $\text{Rule}_\nu$ , where  $\text{Sig}(k) = \tau \mathbf{F}_\nu$  for some  $\tau \in \mathbb{M}$ .
- (B)  $\text{IS\_REWRITABLE}(u, k)$  returns **true**.

The proof follows from inspection of the algorithms that create, inspect, and modify *Rule*.

**Proposition 16.** *If a polynomial multiple  $u\text{Poly}(k)$  is rewritable, then the rewriter  $\text{Poly}(j)$  satisfies  $j > k$ .*

The proof follows from Definitions 11 ( $j' > j$ ) and 14 ( $b < a$ ).

**Proposition 17.** *Let  $k \in G_{\text{curr}}$ . Suppose that a polynomial multiple  $p = u\text{Poly}(k)$  is rewritable by some  $\text{Poly}(j)$  in  $\text{Rule}_\nu$ . If REDUCTION terminates, then there exist  $c \in \mathbb{F}$ ,  $d \in \mathbb{M}$  and  $h_\lambda \in \mathcal{R}$  (for each  $\lambda \in (G_{\text{curr}} \cup \text{completed}) \setminus \{j\}$ ) satisfying*

$$p = cd \cdot \text{Poly}(j) + \sum_{\lambda \in (G_{\text{curr}} \cup \text{completed}) \setminus \{j\}} h_\lambda \text{Poly}(\lambda) \quad (1)$$

where

- for all  $\lambda \in (G_{\text{curr}} \cup \text{completed}) \setminus \{j\}$  if  $h_\lambda \neq 0$  then  $\text{HM}(h_\lambda) \text{Poly}(\lambda)$  has a signature smaller than  $u\text{Sig}(k)$ ; and
- $u\text{Sig}(k)$  is a signature of  $cd \cdot \text{Poly}(j)$ .

*Remark.* It does *not* necessarily follow that  $\mathbf{h}$  is an  $\text{HM}(p)$ -representation of  $p$ . The usefulness of Proposition 17 lies in the fact that all polynomials in (1) have a smaller signature than  $p$  except possibly  $cd \cdot \text{Poly}(j)$ . If  $\text{Poly}(j) = 0$  then the Proposition still holds, since  $u\text{Sig}(k)$  would be a non-minimal signature of the zero polynomial.

*Proof.* Assume that REDUCTION terminates. Let  $\text{Sig}(k) = \tau \mathbf{F}_\nu$ . By Definition 2 there exist  $q_1, \dots, q_\nu \in \mathcal{R}$  such that

$$p = q_1 f_1 + \dots + q_\nu f_\nu,$$

and  $\text{HM}(q_\nu) = \tau$ . Let  $\text{Sig}(j) = \tau' \mathbf{F}_\nu$  and let  $S$  be the  $S$ -polynomial that generated  $\text{Poly}(j)$ . By Definitions 11 and 14, there exist  $H_1, \dots, H_\nu \in \mathcal{R}$  such that

$$S = H_1 f_1 + \dots + H_\nu f_\nu$$

and

- $\text{HM}(H_\nu) = \tau'$ ,
- $\tau' \mid u\tau$ ,
- $\rho = (\tau', j)$  appears in  $\text{Rule}_\nu$ ,
- and  $k \neq j$ .

Let  $\mathcal{G} = G_{\text{curr}} \cup \text{completed}$ . By Definition 11 and the assumption that REDUCTION terminates, there exists  $\mathcal{H} \in \mathcal{R}^{\#\mathcal{G}}$  such that

- $\mathcal{H}$  is a signature-preserving  $S$ -representation of  $S$  w.r.t.  $\{\text{Poly}(\lambda) : \lambda \in \mathcal{G}\}$ ; and
- $\mathcal{H}_j = 1$ .

Let  $d$  be a monomial such that  $d\tau' = u\tau$ . Thus  $d\text{Sig}(j) = u\text{Sig}(k)$ . Let  $\alpha = \text{HC}(h_\nu)$  and  $\beta = \text{HC}(H_\nu)$ . Note that  $\beta \neq 0$ , since it comes from an assigned signature. Then

$$\begin{aligned} p &= \left[ (q_1 f_1 + \cdots + q_\nu f_\nu) - \frac{\alpha}{\beta} dS \right] + \frac{\alpha}{\beta} dS \\ &= \left[ \sum_{\lambda=1}^{\nu} \left( q_\lambda - \frac{\alpha}{\beta} dH_\lambda \right) f_\lambda \right] + \left[ \frac{\alpha}{\beta} d\text{Poly}(j) + \sum_{\lambda \in \mathcal{G} \setminus \{j\}} \left( \frac{\alpha}{\beta} d\mathcal{H}_\lambda \right) \text{Poly}(\lambda) \right] \\ &= \frac{\alpha}{\beta} d \cdot \text{Poly}(j) + \sum_{\lambda \in \mathcal{G} \setminus \{j\}} h_\lambda \text{Poly}(\lambda) \end{aligned} \quad (2)$$

where

$$h_\lambda = \begin{cases} q_\lambda - \frac{\alpha}{\beta} d(H_\lambda - \mathcal{H}_\lambda), & \text{if } \text{Poly}(\lambda) = f_k \text{ for some } k = 1, \dots, \nu; \\ \frac{\alpha}{\beta} d\mathcal{H}_\lambda & \text{otherwise.} \end{cases}$$

Recall that

$$\text{HM}(q_\nu) = u\tau = \text{HM}\left(\frac{\alpha}{\beta} d \cdot H_\nu\right)$$

and since  $\mathcal{H}$  is signature-preserving

$$\text{HM}\left(\frac{\alpha}{\beta} d \cdot \mathcal{H}_\lambda\right) \text{Sig}(\lambda) \prec d\tau' \mathbf{F}_\nu = u\tau \mathbf{F}_\nu \quad \forall \lambda \in \mathcal{G} \setminus \{j\}.$$

Thus for any  $\lambda \in \mathcal{G} \setminus \{j\}$  if  $h_\lambda \neq 0$  then  $\text{HM}(h_\lambda) \text{Sig}(\lambda) \prec u\text{Sig}(k)$ . Recall that  $d\text{Sig}(j) = u\text{Sig}(k)$ . Let  $c = \alpha/\beta$ ; then equation (2) satisfies the proposition.  $\square$

We stumbled on Lemma 18 while trying to resolve a question that arose in our study of the pseudocode of (Faugère, 2002) and (Stegers, 2006). Among the criteria that they use to define a *normalized critical pair*, they mention that the signatures of the corresponding polynomial multiples must be different. However, their pseudocodes for `CRITICAL_PAIR` do not check for this! This suggests that they risk generating at least a few critical pairs that are not normalized, but we have found that this does not occur in practice. Why not?

**Lemma 18.** *Let  $k, \ell \in G_{\text{curr}}$  with  $k > \ell$ . Let  $p = \text{Poly}(k)$ ,  $q = \text{Poly}(\ell)$ , and  $u, v \in \mathbb{M}$ . If  $u\text{Sig}(k) = v\text{Sig}(\ell)$ , then  $v\text{Poly}(\ell)$  is rewritable.*

*Proof.* Assume that  $u\text{Sig}(k) = v\text{Sig}(\ell) = \tau \mathbf{F}_\nu$  for some  $\tau \in \mathbb{M}$ ,  $\nu \in \{1, \dots, m\}$ . Since the signature indices are equal at  $\nu$  and  $k > \ell$ ,  $p$  is a rewriting of an  $S$ -polynomial indexed by  $\text{Rule}_\nu$ , so  $(\text{Sig}(k), k)$  appears in  $\text{Rule}_\nu$  after  $(\text{Sig}(\ell), \ell)$  (assuming that  $(\text{Sig}(\ell), \ell)$  appears at all, which it will not if  $\ell = \nu$ ). Hence  $\text{FIND\_REWRITING}(v, \ell) \neq \ell$ ,  $\text{IS\_REWRITABLE}(v, \ell) = \text{true}$ , and  $v\text{Sig}(\ell)$  is rewritable.  $\square$

#### 4.3. New Characterization of a Gröbner Basis.

**Definition 19.** A *syzygy* of  $F$  is some  $\mathbf{H} \in \mathcal{R}^m$  such that  $\mathbf{H} \cdot F = H_1 f_1 + \cdots + H_m f_m = 0$ .

**Proposition 20.** *Suppose that  $\tau\mathbf{F}_\nu$  is a signature of some  $p \in \mathcal{R}$ , and  $\mathbf{h}$  a corresponding  $F$ -representation. If  $\tau\mathbf{F}_\nu$  is not the minimal signature of  $p$ , then there exists a syzygy  $\mathbf{H}$  of  $F$  satisfying each of the following:*

- (A)  $\tau\mathbf{F}_\nu$  is a signature of  $\mathbf{H} \cdot F$ , and
- (B)  $(\mathbf{h} - \mathbf{H})$  is an  $F$ -representation of  $p$  corresponding to the minimal signature.

*Proof.* Assume that  $\tau\mathbf{F}_\nu$  is not the minimal signature of  $p$ . Suppose that  $\tau'\mathbf{F}_{\nu'}$  is the minimal signature of  $p$ . Then  $\nu' \leq \nu$ . By definition of a signature, there exists  $\mathbf{h}' \in \mathcal{R}^m$  such that

$$p = h_1 f_1 + \cdots + h_\nu f_\nu,$$

$\text{HM}(h_\nu) = \tau$ , and  $\text{HM}(h_\lambda) = 0$  for each  $\lambda = \nu+1, \dots, m$ . Likewise, there exists  $\mathbf{h}' \in \mathcal{R}^m$  such that

$$p = h'_1 f_1 + \cdots + h'_{\nu'} f_{\nu'},$$

$\text{HM}(h'_{\nu'}) = \tau'$ , and  $\text{HM}(h'_\lambda) = 0$  for each  $\lambda = \nu'+1, \dots, m$ . Let

$$H_\lambda = \begin{cases} h_\lambda - h'_\lambda, & 1 \leq \lambda \leq \nu' \\ h_\lambda, & \nu' < \lambda \leq \nu \\ 0, & \nu < \lambda < m \end{cases}$$

for each  $\lambda = 1, 2, \dots, m$ ; then

$$0 = p - p = \sum_{\lambda=1}^m H_\lambda f_\lambda.$$

Let  $\mathbf{H} = (H_1, \dots, H_m)$ ; observe that

- $\mathbf{H}$  is a syzygy of  $F$ ;
- $\tau'\mathbf{F}_{\nu'} \prec \tau\mathbf{F}_\nu$  implies that
  - $h_{\nu'+1} - H_{\nu'+1} = \cdots = h_\nu - H_\nu = 0$  and  $\text{HM}(h_{\nu'} - H_{\nu'}) = \text{HM}(h'_{\nu'}) = \tau'$ ;
  - $\text{HM}(H_\nu) = \tau$ , so  $\tau\mathbf{F}_\nu$  is a signature of  $\mathbf{H} \cdot F$ , satisfying (A); so
  - $(\mathbf{h} - \mathbf{H}) \cdot F$  is an  $F$ -representation of  $p$  corresponding to the minimal signature, satisfying (B).

□

Inspection of the algorithms that assign signatures to polynomials shows that F5 *attempts* to assign the minimal signature with respect to  $F$  of each labeled polynomial in  $r$ :

- the signature assigned to each  $f_i$  of the input is  $\mathbf{F}_i$ ;
- the signatures assigned to  $S$ -polynomials are, by Proposition 7, the smallest one can predict from the information known; and

- if top-reduction would increase a polynomial's signature, then `TOP_REDUCTION` generates a new  $S$ -polynomial with that signature, preserving the signature of the current polynomial.

This does not always succeed, but Theorem 21 implies a benefit.

**Theorem 21** (New characterization). *Suppose that iteration  $i$  of `INCREMENTAL_BASIS` terminates with output  $G_{\text{curr}}$ . Let  $\mathcal{G} = (\text{Poly}(\lambda) : \lambda \in G_{\text{curr}})$ . If every  $S$ -polynomial  $S$  of  $\mathcal{G}$  satisfies (A) or (B) where*

(A)  $S$  has a signature-preserving  $S$ -representation with respect to  $\mathcal{G}$ ;

(B) a component  $u\text{Poly}(k)$  of  $S$  satisfies

(B1)  $u\text{Sig}(k)$  has signature index  $i$  but is not the minimal signature of  $u\text{Poly}(k)$ ;  
or

(B2)  $u\text{Sig}(k)$  is rewritable in *Rule*;

then  $\mathcal{G}$  is a Gröbner basis of  $\langle F_i \rangle$ .

*Remark.* Faugère and Stegers prove a theorem similar to that of Theorem 21 (Theorem 1 in (Faugère, 2002); Theorem 3.21 in (Stegers, 2006)), but their formulation of the theorem does not consider (B2), and their notion of a component's not being “normalized” is less general and not quite the same as (B1).

*Proof.* Let  $S$  be any  $S$ -polynomial of  $\mathcal{G}$ ; say  $S = \sigma_{p,q}p - \sigma_{q,p}q$  where  $p = \text{Poly}(k)$  and  $q = \text{Poly}(\ell)$ . Let  $t = \text{lcm}(\text{HM}(p), \text{HM}(q))$ ; we have  $\text{HM}(\sigma_{p,q}p) = \text{HM}(\sigma_{q,p}q) = t$ . We outline below an iterative process of rewriting those polynomials of the  $\mathcal{G}$ -representation of  $S$  whose head monomials are not smaller than  $t$ . Some rewritings may introduce into the  $\mathcal{G}$ -representation new polynomials whose head monomials are also not smaller than  $t$ . We call both  $S$  and these monomial multiples of  $S$ -polynomials “intermediate  $S$ -polynomials”.

While intermediate  $S$ -polynomials exist in the  $\mathcal{G}$ -representation of  $S$ :

1. Let  $S'$  be the intermediate  $S$ -polynomial with a component  $u\text{Poly}(a)$  that has maximal signature among all components of intermediate  $S$ -polynomials.
  - If  $S'$  satisfies (A), use a signature-preserving  $S$ -representation of  $S'$  to rewrite  $S$  and obtain a new  $\mathcal{G}$ -representation of  $S$ .
  - If  $u\text{Poly}(a)$  satisfies (B1), Proposition 20 implies the existence of a syzygy that rewrites the  $F$ -representation of  $u\text{Poly}(a)$  to one that corresponds to its minimal signature. Use this syzygy to rewrite  $u\text{Poly}(a)$  and obtain a new  $\mathcal{G}$ -representation of  $S$ .
  - Otherwise,  $u\text{Poly}(a)$  satisfies (B2). Choose the rewriter of  $u\text{Poly}(a)$  of maximal index in *Rule* to rewrite  $u\text{Poly}(a)$  in the form indicated by Proposition 17 and obtain a new  $\mathcal{G}$ -representation of  $S$ .
2. Is this new  $\mathcal{G}$ -representation of  $S$  an  $S$ -representation with respect to  $\mathcal{G}$ ? If so, stop. If not, there exist intermediate  $S$ -polynomials in the  $\mathcal{G}$ -representation of  $S$ . Return to step one.



We claim that the iterative process outlined above terminates with an  $S$ -representation of  $S$ . Define

- $\mathcal{A}$ , the set of components of intermediate  $S$ -polynomials of  $\mathcal{G}$  that satisfy (A);
- $\mathcal{B}_1$ , the set of components of intermediate  $S$ -polynomials of  $\mathcal{G}$  that satisfy (B1); and
- $\mathcal{B}_2$ , the set of components of intermediate  $S$ -polynomials of  $\mathcal{G}$  that satisfy (B2).

In addition, define

- $\mathcal{M} = \max_{\prec} \{u\text{Sig}(a) : u\text{Poly}(a) \in \mathcal{A} \cup \mathcal{B}_1 \cup \mathcal{B}_2\}$ ;
- $\mathcal{N} = \max_{\prec} \{u\text{Sig}(a) : u\text{Poly}(a) \in \mathcal{B}_2\}$ ; and
- $\mathcal{O} = \max_{<_T} \{\text{HM}(u\text{Poly}(a)) : u\text{Poly}(a) \in \mathcal{A}\}$ .

After each iteration,  $\mathcal{N} \leq \mathcal{M}$  remains invariant, and one of the following occurs.

- After an intermediate  $S$ -polynomial satisfying (A) is rewritten, the signature-preserving representation guarantees that any component of a newly introduced intermediate  $S$ -polynomial has a signature smaller than  $u\text{Sig}(a)$ , except possibly one,  $d\text{Sig}(j)$  for some  $j \in G_{\text{curr}}$  and some  $d \in \mathbb{M}$ . By Definition 9  $b > a$ . Thus  $\mathcal{O}$  decreases, and  $\mathcal{M}$  does not increase.
- After a component  $u\text{Poly}(a)$  satisfying (B1) is rewritten, by Proposition 20 the signatures of components of newly introduced intermediate  $S$ -polynomials are smaller than  $u\text{Sig}(a)$ . Since  $u\text{Poly}(a)$  was chosen to have maximal signature,  $\mathcal{M}$  decreases.
- After a component satisfying (B2) is rewritten, by Proposition 17 only the signature associated with the rewriter  $d\text{Poly}(j)$  has value  $u\text{Sig}(a)$ , for some  $j \in G_{\text{curr}}$  and some  $d \in \mathbb{M}$ . By Proposition 16  $j > a$ . We chose the rewriting with maximal index in  $\text{Rule}$  to rewrite this signature, so if  $d\text{Poly}(j)$  is a component of a newly-introduced intermediate  $S$ -polynomial, then it does not satisfy (B2); that is,  $d\text{Poly}(j)$  is not itself rewritable. Thus  $\mathcal{M}$  does not increase, and since  $u\text{Poly}(a)$  was chosen to have maximal signature,  $\mathcal{N}$  decreases.

After each rewriting, one of  $\mathcal{M}$ ,  $\mathcal{N}$ , or  $\mathcal{O}$  decreases. Observe that  $\mathcal{M}$  never increases. If  $\mathcal{N}$  increases (as it can during an (A) or (B1) rewriting), its new value is no larger than that of  $\mathcal{M}$  before the rewriting. If  $\mathcal{O}$  increases (as it can during a (B1) or (B2) rewriting) then one of  $\mathcal{M}$  or  $\mathcal{N}$  decreases. Thus the only possibility for an infinite loop is the case where  $\mathcal{N}$  decreases while increasing  $\mathcal{O}$ , then  $\mathcal{O}$  decreases while returning  $\mathcal{N}$  to its previous value. This cannot continue indefinitely, because both (A) and (B2) rewritings increases the index in  $r$  of the polynomial having signature  $u\text{Sig}(a)$  (since  $b > a$  and  $j > a$ ) and  $r$  has only finitely many elements. Along with the well-ordering property common to both  $\prec$  and  $<_T$ , this implies that the iterative process terminates with an  $S$ -representation of  $S$ .  $\square$

#### 4.4. Principal Syzygies

Suppose that all syzygies of  $F$  are generated by principal syzygies of the form  $f_i \mathbf{F}_j - f_j \mathbf{F}_i$ . If  $\text{Sig}(k)$  is not minimal, then by Proposition 20 some monomial multiple of a principal syzygy  $\mu(f_i \mathbf{F}_j - f_j \mathbf{F}_i)$  has the same signature as  $\text{Sig}(k)$ . This provides an easy test for such a non-minimal signature.

**Definition 22.** We say that a polynomial multiple  $u\text{Poly}(k)$  satisfies Faugère's criterion with respect to  $G_{\text{prev}}$  if

- $\text{Sig}(k) = \tau \mathbf{F}_\nu$ ; and
- there exists  $\ell \in G_{\text{prev}}$  such that
  - $\text{Sig}(\ell) = \tau' \mathbf{F}_{\nu'}$  where  $\nu' < \nu$ ; and
  - $\text{HM}(\text{Poly}(\ell))$  divides  $u\tau$ .

**Proposition 23.** If a polynomial multiple  $u\text{Poly}(k)$  satisfies Faugère's criterion with respect to  $G_{\text{prev}}$  then  $u\text{Sig}(k)$  is not the minimal signature of  $u\text{Poly}(k)$ .

*Proof.* Assume that a polynomial multiple  $u\text{Poly}(k)$  satisfies Faugère's criterion with respect to  $G_{\text{prev}}$ . Let  $p = \text{Poly}(k)$  and  $\tau \mathbf{F}_\nu = \text{Sig}(k)$ , so there exists  $\mathbf{h} \in \mathcal{R}^m$  such that

$$p = h_1 f_1 + \cdots + h_m f_m,$$

$h_{\nu+1} = \cdots = h_m = 0$ , and  $\text{HM}(h_\nu) = \tau$ . From the definition of Faugère's criterion, there exists  $\ell \in G_{\text{prev}}$  such that  $\text{HM}(\text{Poly}(\ell))$  divides  $u\tau$ . Let  $q = \text{Poly}(\ell)$ . Since  $\ell \in G_{\text{prev}}$ , there exists  $\mathbf{H} \in \mathcal{R}^m$  such that  $\nu' < \nu$ ,

$$q = H_1 f_1 + \cdots + H_m f_m,$$

$H_{\nu'+1} = \cdots = H_m = 0$ , and  $H_{\nu'} \neq 0$ . Choose  $d \in \mathbb{M}$  such that  $d \cdot \text{HM}(q) = u\tau$ . Observe that

$$\begin{aligned} up &= u(h_1 f_1 + \cdots + h_\nu f_\nu) \\ &= u \left[ \sum_{\lambda=1}^{\nu-1} h_\lambda f_\lambda + (h_\nu - \text{HM}(h_\nu)) \cdot f_\nu \right] + u \text{HM}(h_\nu) f_\nu. \end{aligned} \tag{3}$$

Let

$$P = u \left[ \sum_{\lambda=1}^{\nu-1} h_\lambda f_\lambda + (h_\nu - \text{HM}(h_\nu)) \cdot f_\nu \right];$$

equation (3) becomes

$$\begin{aligned} up &= P + u \cdot \text{HM}(h_\nu) f_\nu \\ &= P + (u\tau) \cdot f_\nu \\ &= P + (d \cdot \text{HM}(q)) \cdot f_\nu \\ &= P + d \cdot \left( \text{HM} \left( \sum_{\lambda=1}^{\nu'} H_\lambda f_\lambda \right) \right) \cdot f_\nu. \end{aligned} \tag{4}$$

By the distributive and associative properties

$$\left( \sum_{\lambda=1}^{\nu'} H_{\lambda} f_{\lambda} \right) f_{\nu} = \sum_{\lambda=1}^{\nu'} f_{\lambda} (H_{\lambda} f_{\nu}),$$

so

$$\text{HM} \left( \sum_{\lambda=1}^{\nu'} H_{\lambda} f_{\lambda} \right) f_{\nu} = \sum_{\lambda=1}^{\nu'} f_{\lambda} (H_{\lambda} f_{\nu}) - \left[ \left( \sum_{\lambda=1}^{\nu'} H_{\lambda} f_{\lambda} \right) - \text{HM} \left( \sum_{\lambda=1}^{\nu'} H_{\lambda} f_{\lambda} \right) \right] f_{\nu}.$$

Let

$$Q = \sum_{\lambda=1}^{\nu'} f_{\lambda} (H_{\lambda} f_{\nu}) \quad \text{and} \quad R = \left[ \left( \sum_{\lambda=1}^{\nu'} H_{\lambda} f_{\lambda} \right) - \text{HM} \left( \sum_{\lambda=1}^{\nu'} H_{\lambda} f_{\lambda} \right) \right].$$

We can rewrite equation (4) as

$$up = P + dQ - (dR) \cdot f_{\nu}.$$

We claim that we have rewritten  $up$  with a signature smaller than  $(u\tau) \mathbf{F}_{\nu}$ . By construction,  $P$  has a signature smaller than  $(u\tau) \mathbf{F}_{\nu}$ . By inspection,  $Q$  has a signature index no greater than  $\nu'$ , so  $dQ$  has a signature smaller than  $(u\tau) \mathbf{F}_{\nu}$ . That leaves  $(dR) \cdot f_{\nu}$ , and

$$\begin{aligned} \text{HM}(dR) &= d \cdot \text{HM} \left[ \left( \sum_{\lambda=1}^{\nu'} H_{\lambda} f_{\lambda} \right) - \text{HM} \left( \sum_{\lambda=1}^{\nu'} H_{\lambda} f_{\lambda} \right) \right] \\ &= d \cdot \text{HM}(q - \text{HM}(q)) \\ &<_T d \cdot \text{HM}(q) \\ &= u\tau. \end{aligned}$$

Hence  $(dR) \cdot f_{\nu}$  has a signature smaller than  $(u\tau) \mathbf{F}_{\nu}$ , and  $up$  has a signature smaller than  $(u\tau) \mathbf{F}_{\nu}$ . That is,  $u\text{Sig}(k)$  is not the minimal signature of  $up$ .  $\square$

If a polynomial multiple  $u\text{Poly}(k)$  satisfies Faugere's criterion with respect to  $G_{\text{prev}}$ , then by Proposition 23 and Theorem 21 we need not compute it. `CRITICAL_PAIR` and `FIND_REDUCUTOR` discard any polynomial multiple that satisfies Faugère's criterion. Thus Theorem 21 and Proposition 23 show that:

**Corollary 24.** *Given  $F$ , the output of the F5 algorithm is a Gröbner basis of  $\langle F \rangle$ . Also, if all the syzygies of  $F$  are principal, then F5 does not reduce any polynomials to zero.*

Corollary 24 does *not* imply:

- that F5 does not generate redundant polynomials. The example from (Faugère, 2002) generates one such polynomial ( $r_{10}$ ).
- that F5 terminates, at least not obviously. To the contrary, `FIND_REDUCUTOR` rejects potential reducers that are rewritable or that satisfy Faugère's criterion. As a result, the algorithm can compute a Gröbner basis, while new polynomials that are not completely top-reduced continue to generate new critical pairs. We have not observed this in practice.

#### 4.5. Correctness of the output of F5C

We come now to the correctness of F5C. For correctness, we argue that each stage of F5C imitates the behavior of F5 on an input equivalent to the data structures generated by `SETUP_REDUCED_BASIS`. Recall that  $F_i = (f_1, \dots, f_i)$ . We will refer to the system  $F' = (B_1, \dots, B_{\#B}, f_{i+1})$  where

- $B$  is computed during the execution of `SETUP_REDUCED_BASIS`; and
- $F'$  is indexed as  $F'_i = B_i$ , etc.

It is trivial that  $\langle B \rangle = \langle F_i \rangle$  and  $\langle F' \rangle = \langle F_{i+1} \rangle$ .

**Lemma 25.** *When `SETUP_REDUCED_BASIS` terminates, every element of  $r$  is admissible with respect to  $B$ , and thus with respect to  $F'$ .*

The proof is evident from inspection of `SETUP_REDUCED_BASIS`.

The correctness of the behavior of `IS_REWRITABLE` in F5C hinges on Definition 26.

**Definition 26.** Let  $t \in \mathbb{M}$  and  $k \in G_{\text{curr}}$ . At any point in the algorithm, we say that a polynomial multiple  $t\text{Poly}(k)$  is *rewritable by the zero polynomial* if there exist  $a, b \in G_{\text{prev}}$  such that

- the  $S$ -polynomial  $S$  of  $p = \text{Poly}(a)$  and  $q = \text{Poly}(b)$  reduces to zero, although the reduction may not be signature-safe; and
- $\max(\sigma_{p,q}\text{Sig}(a), \sigma_{q,p}\text{Sig}(b))$  divides  $t\text{Sig}(k)$ .

*Remark.* It is essential that  $a, b \in G_{\text{prev}}$  and not in  $G_{\text{curr}}$ . The fact that a component of an  $S$ -polynomial is rewritable does *not* imply that it is rewritable by the zero polynomial. Proposition 17 implies that if a component of an  $S$ -polynomial is rewritable, then the  $S$ -polynomial can be rewritten using a polynomial of the same signature; *however*, the resulting  $S$ -representation may not yet exist when the component is detected to be rewritable.

**Lemma 27.** *When `SETUP_REDUCED_BASIS` terminates, `IS_REWRITABLE` in F5C would return `true` for the input  $(u, k)$ , only if  $u\text{Poly}(k)$  is rewritable by the zero polynomial.*

*Proof.* Line 7 of `SETUP_REDUCED_BASIS` interreduces the polynomials indexed by  $G_{\text{prev}}$  to obtain the reduced Gröbner basis  $B$ . Thus all  $S$ -polynomials of  $B$  are rewritable by the zero polynomial. When `SETUP_REDUCED_BASIS` terminates, *Rule* consists of a list of lists. Elements of the  $j$ th list have the form  $\omega_k = (\sigma_{B_j, B_k}, 0)$  for  $k = 1, \dots, j-1$  where, as explained in the introduction,

$$\sigma_{B_j, B_k} = \frac{\text{lcm}(\text{HM}(B_j), \text{HM}(B_k))}{\text{HM}(B_j)}.$$

Thus if `IS_REWRITABLE`( $u, k$ ) is true, then  $u\text{Poly}(k)$  is rewritable by the zero polynomial.  $\square$

**Corollary 28.** *In F5C, if `IS_REWRITABLE` returns `true` for the input  $(u, k)$ , then  $u\text{Poly}(k)$  is rewritable either by a polynomial that appears in  $r$ , or by the zero polynomial.*

*Proof.* This is evident from consequence of Proposition 15 and the isolation of all modifications of F5 to `SETUP_REDUCED_BASIS`. Let  $j = \text{FIND\_REWRITING}(u, k)$ . If  $j = 0$ , Lemma 27 implies that  $u\text{Poly}(k)$  is rewritable by the zero polynomial. Otherwise, line 10 of `SETUP_REDUCED_BASIS` implies that  $j \in G_{\text{curr}} \setminus G_{\text{prev}}$ . That is,  $r_j$  was generated in the same way that F5 would generate it. By Proposition 15,  $u\text{Poly}(k)$  is rewritable by  $\text{Poly}(j)$ .  $\square$

**Theorem 29.** *If `INCREMENTAL_BASIS/C` terminates for a given input  $i$ , then it terminates with a Gröbner basis of  $\langle F_i \rangle$ .*

*Proof.* The proof is adapted easily from the proof of Theorem 21, using Lemma 25 and Corollary 28. In particular,  $S$ -polynomials that are rewritable by the zero polynomial—that is, the  $S$ -polynomials of  $B$ —can be rewritten in the same manner as polynomials that satisfy case (A) of Theorem 21.  $\square$

Changing the algorithm’s point of view so that some polynomials are admissible with respect to  $F'$  and not to  $F$  implies the possibility of introducing non-principal syzygies. Of course we would like F5C to avoid any reductions to zero that F5 also avoids; otherwise the benefit from a reduced Gröbner basis could be offset by the increased cost of wasted computations. Hence we must show that if the syzygies of the input  $F$  are all principal, then F5C does not introduce reductions to zero. Lemma 30 shows that the signature of a polynomial indexed by  $G_{\text{curr}} \setminus G_{\text{prev}}$  in F5C corresponds to the signature that F5 would compute, “translated” by  $\#B - (i - 1)$ .

**Lemma 30.** *Let  $i > 2$ . During the  $i$ th pass through the while loop of `BASIS/C`, let  $k \in G_{\text{curr}} \setminus G_{\text{prev}}$ , and  $\text{Sig}(k) = \tau \mathbf{F}_\nu$ , where the signature is with respect to  $F'$ . Then  $(\tau \mathbf{F}_i, \text{Poly}(k))$  is admissible with respect to  $F$ .*

*Proof.* From the assumption that  $\text{Sig}(k) = \tau \mathbf{F}_\nu$ , we know that there exists  $\mathbf{h} \in \mathcal{R}$  such that

$$\text{Poly}(k) = \sum_{\lambda=1}^m h_\lambda F'_\lambda,$$

$h_{\nu+1} = \dots = h_m = 0$ , and  $\text{HM}(h_\nu) = \tau$ . Recall that  $F'_\lambda = B_\lambda$  for each  $\lambda = 1, \dots, \nu - 1$ . By Theorem 29, there exist  $H_1, \dots, H_{i-1}$  such that

$$\sum_{\lambda=1}^{\nu-1} h_\lambda F'_\lambda = \sum_{\lambda=1}^m H_\lambda f_\lambda$$

and  $H_\nu = \dots = H_m = 0$ . In addition,  $F'_\nu = f_i$ . Hence

$$\text{Poly}(k) = \sum_{\lambda=1}^{\nu-1} H_\lambda f_\lambda + h_\nu f_i,$$

whence  $(\tau \mathbf{F}_i, \text{Poly}(k))$  is admissible with respect to  $F$ .  $\square$

**Theorem 31.** *If the syzygies of  $F$  are all principal syzygies, then F5C does not reduce any polynomial to zero.*

*Proof.* Assume for the contrapositive that  $k \in G_{\text{curr}}$  and the algorithm reduces  $\text{Poly}(k)$  to zero. Suppose that we are on iteration  $i$  of the **while** loop of BASIS/C. Let  $\text{Sig}(k) = \tau \mathbf{F}_i$ . This signature of  $\text{Poly}(k)$  is with respect to  $F'$ ; from Lemma 30 we infer that  $\tau \mathbf{F}_i$  is a signature of  $\text{Poly}(k)$  with respect to  $F$ .

Now  $G_{\text{prev}}$  indexes a reduced Gröbner basis  $B$  of  $\langle F_i \rangle$ . The reduction to zero implies that CRITICAL\_PAIR did not discard the corresponding critical pair, which in turn implies that no head monomial of  $B$  divided  $\tau$ . By the definition of a reduced Gröbner basis, no head monomial of the unreduced basis would have divided  $\tau$  either. By Corollary 24, the syzygies of  $F$  are not principal.  $\square$

*Remark.* In our experiments with inputs whose syzygies are not principal, it remains the case that F5C computes no more reductions to zero than does F5. However, we do not have a proof of this. The difficulty lies in the fact that signatures of polynomials in  $G_{\text{prev}}$  need not be the same in F5 and F5C. F5 computes different critical pairs, which may generate different rewrite rules. This introduces the possibility that F5 rejects some polynomials as rewritable that F5C does not. However, we have not observed this in practice.

We conclude with two final, surprising results.

**Theorem 32.** *In SETUP\_REDUCED\_BASIS, there is no need to recompute the rewrite rules for  $B$ .*

*Proof.* When performing top-reductions by elements of  $B$ , the algorithm checks neither whether a polynomial multiple is rewritable, nor whether it satisfies Faugère's criterion. Thus we only need to verify the statement of the theorem in the context of  $S$ -polynomial creation. Suppose therefore that we are computing  $G_i$ , the Gröbner basis of  $\langle F'_i \rangle$  where  $F'_i = (B_1, \dots, B_{\#B}, f_i)$ , and while computing the  $S$ -polynomial of  $p = \text{Poly}(k)$  and  $q = \text{Poly}(\ell)$ , where  $k \in G_{\text{curr}} \setminus G_{\text{prev}}$  and  $\ell \in G_{\text{prev}}$ , IS\_REWRITABLE reports that  $\sigma_{q,p}q$  is rewritable.

We claim that it will also reject  $\sigma_{p,q}p$ . Since  $\sigma_{q,p}q$  is rewritable, there exists  $j \in \{1, \dots, \#B\}$  such that

$$\frac{\text{lcm}(\text{HM}(q), (\text{HM}(B_j)))}{\text{HM}(q)} \text{ divides } \frac{\text{lcm}(\text{HM}(p), \text{HM}(q))}{\text{HM}(q)}.$$

It follows that  $\text{lcm}(\text{HM}(q), \text{HM}(B_j))$  divides  $\text{lcm}(\text{HM}(p), \text{HM}(q))$ . A straightforward argument on the degrees of the variables implies that  $\text{lcm}(\text{HM}(p), \text{HM}(B_j))$  also divides  $\text{lcm}(\text{HM}(p), \text{HM}(q))$ . Thus

$$\frac{\text{lcm}(\text{HM}(p), \text{HM}(B_j))}{\text{HM}(p)} \text{ divides } \frac{\text{lcm}(\text{HM}(p), \text{HM}(q))}{\text{HM}(p)} = \sigma_{p,q}.$$

The design of the algorithm implies that the  $S$ -polynomial of  $p$  and  $B_j$  would have been considered before the  $S$ -polynomial of  $p$  and  $q$ . This leads to two possibilities.

1. The  $S$ -polynomial of  $p$  and  $B_j$  was computed, so that the rewrite rule  $(\sigma_{p,B_j}, \lambda)$  appears in  $\text{Rule}_i$  for some  $\lambda \in G_{\text{curr}}$ . Hence  $\text{IS\_REWRITABLE}(\sigma_{p,q}, k)$  returns **true**.
2. The  $S$ -polynomial of  $p$  and  $B_j$  was rejected, either because  $\sigma_{p,B_j}p$  is rewritable or because it satisfies Faugère's criterion. Either one implies that the  $\sigma_{p,q}p$  will also be rejected.

Hence there is no need to compute the rewrite rules for  $B$ .  $\square$

**Corollary 33.** *We can reformulate F5C so that `SETUP_REDUCED_BASIS` is unnecessary, and the list `Rule` records only signatures of polynomials indexed by  $G_{\text{curr}} \setminus G_{\text{prev}}$ .*

*Proof.* Theorem 32 implies that we do not need the signatures of polynomials indexed by  $G_{\text{prev}}$  for the rewrite rules. In fact, this is the only reason we might need their signatures, since `COMPUTE_SPOLS` always uses the larger signature to create an  $S$ -polynomial, and `TOP_REDUCTION` top-reduces by  $B$  without checking signatures. Hence the signatures of polynomials indexed by  $G_{\text{prev}}$  are useless. We now indicate how to revise the algorithm to take this into account.

As in F5R, replace line 22 of `BASIS` with

22    Let  $B$  be the interreduction of  $\{\text{Poly}(\lambda) : \lambda \in G_{\text{prev}}\}$ .

Subsequently, change line 17 of `CRITICAL_PAIR` to

17    **if**  $k \notin G_{\text{prev}}$  **and**  $u_1 \cdot \tau_1$  is top-reducible by  $G_{\text{prev}}$

and line 19 of `CRITICAL_PAIR` to

19    **if**  $\ell \notin G_{\text{prev}}$  **and**  $u_2 \cdot \tau_2$  is top-reducible by  $G_{\text{prev}}$

Similarly adjust line 10 of `COMPUTE_SPOLS` and line 16 of `FIND_REDUCUTOR` so that they do not check polynomials of  $G_{\text{prev}}$ . Modify the definition of `Rule` so that it is only one list, not a list of lists, and `FIND_REWRITING` so that it only searches backwards through `Rule`, rather than finding which list in `Rule` to check. Theorem 32 implies that if the original F5C terminates correctly, then this modified version of F5C also terminates correctly.  $\square$

*Remark.* Theorem 32 applies only to F5C, not to F5. The difference is that for any  $\ell \in G_{\text{prev}}$ , F5C guarantees that  $\text{Sig}(\ell) = \tau \mathbf{F}_\ell$  where  $\tau = 1$ . This is not the case in F5.

The prototype implementations of F5C are primarily for educational purposes, so for the sake of clarity we implement the given pseudocode without the optimization outlined in the proof of Corollary 33.

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Figure 1: Example run of the Singular prototype system

```

> LIB "f5_library.lib";
// ** loaded f5_library.lib (1.1,2009/01/26")
> ring R = 0,(x,y,z,t),dp;
> ideal i = yz3 - x2t2, xz2 - y2t, x2y - z2t;
> ideal B = basis(i);
Iteration 2
Processing 1 critical pairs of degree 5
Processing 1 critical pairs of degree 7
4 polynomials in basis
Iteration 3
Processing 1 critical pairs of degree 5
Processing 1 critical pairs of degree 6
Processing 4 critical pairs of degree 7
Processing 1 critical pairs of degree 8
10 polynomials in basis

number of zero reductions: 0
number of elements in g: 10
cpu time for gb computation: 50/1000 sec
> B;
B[1]=yz3-x2t2
B[2]=x2y-z2t
B[3]=xz2-y2t
B[4]=xy3t-z4t
B[5]=z6t-y5t2
B[6]=y3zt-x3t2
B[7]=z5t-x4t2
B[8]=y5t2-x4zt2
B[9]=x5t2-y2z3t2
B[10]=y6t2-xy2zt4
>

```



## Appendix: Using the SINGULAR and Sage prototype implementations

The SINGULAR prototype implementation contains three functions `basis`, `basis_r`, and `basis_c` to compute the Gröbner basis of an ideal. An example run with `basis` is shown in Figure 1. While computing the Gröbner basis, this implementation also prints for each degree the size of  $P_d$ , the set of critical pairs passed to `COMPUTE_SPOLS`. This implementation checks in both `CRITICAL_PAIR` and `COMPUTE_SPOLS` for the rewritten criterion, so  $\#P_d$  is sometimes smaller here than in Faugère's paper, but the reader can compare the results to see that the same basis is generated. A large number of benchmark systems can be obtained by downloading the companion file

<http://www.math.usm.edu/perry/Research/f5ex.lib> .

For a further introduction to SINGULAR, see (Greuel and Pfister, 2008).

The Sage prototype implementation contains four classes, `F5`, `F5R`, `F5C`, and `F4F5`. These can be called by creating the appropriate class with a Sage ideal. An example run with `F4F5` is shown in Figure 2. As in the SINGULAR implementation, run-time data is printed. In this case, the number of critical pairs in  $P_d$ , the number of polynomials generated by `COMPUTE_SPOLS`, and the size of the matrix used for Gaussian elimination. No special techniques are used for sparse matrices in this version, so it is rather slow (in fact, it is slower than the other `F5`'s). The reader should notice that in this version, the output has been interreduced, so there are only 8 polynomials in the final result. For more information on Sage, visit

<http://www.sagemath.org/> .

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Figure 2: Example run of the Sage prototype implementation

```

sage: attach "/home/perry/common/Research/SAGE_programs/f5.py"
sage: f5 = F4F5()
sage: R.<x,y,z,t> = QQ[]
sage: I = R.ideal(y*z^3-x^2*t^2, x*z^2 - y^2*t, x^2*y - z^2*t)
sage: B = f5(I)
Increment 1
1 critical pairs
Processing 1 pairs of degree 5 of 1 total
1 polynomials generated
1 x 2, 1, 0
1 polynomials left
Processing 1 pairs of degree 7 of 1 total
1 polynomials generated
1 x 2, 1, 0
1 polynomials left
Ended with 4 polynomials
Increment 2
4 critical pairs
Processing 1 pairs of degree 5 of 4 total
1 polynomials generated
1 x 2, 1, 0
1 polynomials left
Processing 2 pairs of degree 6 of 6 total
1 polynomials generated
1 x 2, 1, 0
1 polynomials left
Processing 4 pairs of degree 7 of 6 total
2 polynomials generated
4 x 6, 4, 0
2 polynomials left
Processing 2 pairs of degree 8 of 2 total
1 polynomials generated
2 x 3, 2, 0
1 polynomials left
Ended with 10 polynomials
sage: B
[x*z^2 - y^2*t,
x^2*y - z^2*t,
x*y^3*t - z^4*t,
y*z^3 - x^2*t^2,
y^3*z*t - x^3*t^2,
z^5*t - x^4*t^2,
y^5*t^2 - x^4*z*t^2,
x^5*t^2 - z^2*t^5]
sage:

```

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